MATH 147 QUIZ 3 SOLUTIONS

1. State the Second Derivative Test for the function $f(x, y)$ with critical point (a, b). (2 Points)

   Given $f(x, y)$ with first and second partial derivatives existing, and continuous around an open disk centered at (a, b), the second derivative test is as follows. For

   $$A = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b),$$

   we have the following cases:

   (a) If $A > 0$ and $f_{xx} > 0$, the function has a relative minimum at (a, b).

   (b) If $A > 0$ and $f_{xx} < 0$, the function has a relative maximum at (a, b).

   (c) If $A < 0$, the function has a saddle point at (a, b).

   (d) If $A = 0$, the test in inconclusive.

2. Find and classify the critical points of $f(x, y) = x^2 + x - 3xy + y^3 - 5$. (8 points)

   To find the critical points, we take the first partial derivatives.

   $$f_x = 2x + 1 - 3y,$$
   $$f_y = -3x + 3y^2.$$

   We set these equal to zero to find the critical points. Note that these exist for all $(x, y) \in \mathbb{R}^2$, so these are all the critical points. Solve the system of equations (via substitution or elimination) to get the critical points $p_1 = (1, 1)$ and $p_2 = (1/4, 1/2)$.
   Next we classify these using the second derivative test. As these are continuous everywhere, we need to find only f_{xx}, f_{yy}, and f_{xy}. These are

   $$f_{xx} = 2$$
   $$f_{yy} = 6y$$
   $$f_{xy} = -3.$$

   Then, we calculate A. Note that for both points, $f_{xx} = 2$ and $f_{xy}^2 = 9$. At $(1, 1)$, $f_{yy}(1, 1) = 6$, so

   $$A(p_1) = (2)(6) - 9 = 3.$$

   As A is positive and f_{xx} is positive, we must have a relative minimum at $(1, 1)$. On the other hand, at $(1/4, 1/2)$, $f_{yy}(1/4, 1/2) = 3$. Then we get

   $$A(p_2) = (2)(3) - 9 = -3.$$

   And as $A < 0$, we have a saddle point at $(1/4, 1/2)$.